AN EXAMPLE IN THE THEORY OF STABLE MARKOV OPERATORS

BY

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ABSTRACT

An example of a Markov transtion operator acting on $C(X)$ is constructed in which the restriction of the operator to the conservative part of X is not a conservative operator.

Let $C(X)$ denote, as usual, the B-space of continuous real-valued functions on the compact Hausdorff space X. A (stable) Markov operator on $C(X)$ is a non-negative linear operator T with $T1 = 1$. In his development of a topological analogue of the Hopf ergodic decomposition for such an operator, Foguel [1] defines the following.

Let $A = \{f\colon\ 0 \leqq f \leqq 1,\ f$ is l.s.c., $Tf \leqq f$, and $T^n f \downarrow 0$ (pointwise)$\}$. Let $D(f) = \{x\colon\ f(x) > 0\}$ and then define the dissipative set as $D = \cap\{D(f) : f \in A\}$. The conservative set, $C = X \setminus D$, is then closed and invariant.

Since C is closed and invariant, the process may be restricted to the set C to obtain a new (stable) Markov operator acting on C. Foguel defines a conservative Markov operator to be one for which D is empty, thus avoiding the open question of whether T restricted to C is conservative. Conditions in terms of catagory have been given by Lin [2] for this to be the case.

First we let I be the unit interval $[0,1]$ and consider the point transformation $\phi\colon x \to 2x \pmod 1$. The Markov operator which is generated by ϕ is not stable on $C(I)$, of course. But now we look at the orbit of the point $p = 0.1010010001 \cdots$ (dyadic) under the action of the iterates ϕ^n. The discontinuity of ϕ at $w = 0.1000\cdots$ is only approached from above by the iterates. Thus ϕ acts continuously on $C(S)$ where S is the closure of the set of iterates of p. It is easily seen that S is countable

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and the only limit points of the set of iterates are the points of the form $W_n = 2^{-n}$ and the origin.

Now we take the unit square $Z = I \times I$ and define the point transformation $\theta: (1/2x, y) \to (x, \phi(y))$ where ϕ is the map discussed above. Let $w = (1, p)$ where $p \in I$ is the point discussed above. Let X be the closure of the orbit of w under the action of θ. As before, θ is continuous on X as the segment $\{[x, 1/2] : x \in I\}$, which is the discontinuity set of θ, is only approached from above by the iterates of w. The limit points of the iterates are countable set $(0, 2^{-n})$ and $(0,0)$. Thus θ generates a Markov operator which acts on $C(X)$.

We wish to compute the conservative set for (X, θ). Note that for $F(x, y) = x$, we have $F \in A$ so that the conservative part is contained in $\{(0, y) : y \in I\} \cap X$. But now consider any function, $0 \leqq f \leqq 1$, which is lower-semicontinuous on X. If some point u of $\{(0, y) : y \in I\} \cap X$ is in the open set $[f > 0|$, then it is in the open set $[f > 1/2f(u)]$ as well. Now u is a cluster point of $\theta^n w$ so $T^n f(w) > 1/2f(u) > 0$ for infinitely many n. Thus $T^n f \downarrow 0$ is not possible, so f is not in A. Thus the conservative part of $(\boldsymbol{X}, \theta)$ is the set $Y = \{(0, y) : y \in I\} \cap X$.

Finally, we consider the Markov operator induced by restriction to Y. Let $G(x, y) = 1 - H(x,y)$, where $H(x, y)$ is the indicator function of the point $(0,0)$. Then B is seen to satisfy $T^n G \downarrow 0$, where T denotes here the restricted operator. Thus the conservative part of (Y, θ) is singleton point $(0,0)$. Therefore the conservative part of (X, θ) is not conservative.

References

1. S. R. Foguel, *Ergodic decompositions of a topological space*, Israel J. Math. **7** (1969), 164–167.

2. M. Lin, *Conservative Markov processes on a topological space*, Israel J. Math. **8** (1970), 165–186.

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